

## PRE-SERVICE TEACHERS' MEANINGS AND NON-CANONICAL GRAPHS

Kevin C. Moore  
University of Georgia  
kvcmoore@uga.edu

David R. Liss II  
University of Georgia  
drliss@uga.edu

Jason Silverman  
Drexel University  
js657@drexel.edu

Teo Paoletti  
University of Georgia  
paolett2@uga.edu

Kevin R. LaForest  
University of Georgia  
laforesk@uga.edu

Stacy Musgrave  
University of Georgia  
smusgrav@uga.edu

*Research continues to illustrate the important role of covariational and quantitative reasoning in the context of function and graphing. The same body of literature has emphasized that students and teachers often construct meanings for function and graphing that do not foreground these reasoning processes. In order to gain deeper insights into such meanings, we conducted clinical interviews with ten pre-service secondary teachers. In the present work, we illustrate the construct of shape thinking in relation to their graphing activity during the clinical interviews. We draw particular attention to the implications of shape thinking, including constraints generated by meanings rooted in such thinking, when confronted with non-canonical situations.*

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Students' function meanings remains a critical area of research in mathematics education (Oehrtman, Carlson, & Thompson, 2008), with a growing body of literature characterizing the role of covariational and quantitative reasoning in supporting students' function concept (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Castillo-Garsow, 2012; Ellis, 2007; Moore, 2012; Thompson, 1994, 2011; Weber, 2012). For instance, Castillo-Garsow (2012) identified that the manner in which students' conceive quantities and how they change in tandem has serious implications for their images of exponential growth. Likewise, Moore (2012) and Weber (2012) illustrated the importance of students' attention to dynamic relationships between quantities in the context of trigonometric functions and two-variable functions, respectively.

Although it is apparent that quantitative and covariational reasoning are important for students' function concept, students' mathematical experiences typically lack a fundamental focus on such reasoning (Oehrtman et al., 2008; Smith III & Thompson, 2008; Thompson, in press). In turn, these students construct function meanings devoid of imagery that involves varying quantities' values. As cases in point, Thompson (1994), Goldenberg and colleagues (1992), and Weber (2012) documented that students have a tendency to conceive graphs as pictorial objects with various global properties that are not grounded in quantitative reasoning. Weber and Thompson termed such ways of thinking as *shape thinking* (Weber, 2012).

We extend the shape thinking construct by exploring pre-service secondary mathematics teachers' (PSTs') activity during clinical interviews (Goldin, 2000) designed to offer insights into their meanings, particularly in the context of graphing. Based on several classroom events that suggested PSTs engage in shape thinking, we designed interview tasks such that shape thinkers might face perturbations in their meanings. By generating such situations, we characterize shape thinkers' meanings, including constraints shape thinkers face when confronted with situations that are not supportive of shape thinking. Against the backdrop of our results, we discuss several implications of shape thinking and provide conjectures about the nature of students' mathematical experiences that contribute to shape thinking.

### Theoretical Framing

When one speaks of meanings, he or she is speaking of a pervasive, yet complex and often ill-defined term (Thompson, in press). Thus, it is necessary that we provide a brief description of our use of the term *meaning*. Drawing on the works of several individuals who concerned themselves with epistemology

(e.g., John Dewey, Ernst von Glasersfeld, and Jean Piaget), we consider meanings to be constructions unique to an individual that organize her experiences. As Thompson (in press) described, and drawing on Piaget's description of understanding and meaning, the act of constructing meaning entails assimilation; to construct a meaning is to construct a scheme through repeated reasoning that enables one to organize her experiences.

Due to the assimilatory nature of meanings, meanings influence how an individual makes sense of her future experiences. But, we caution that claiming meanings are assimilatory does not imply that meanings are static cognitive structures to be applied as is. Rather, meanings become more and more stable through an individual repeatedly reconstructing these meanings to make sense of her experiences (Thompson, in press). Meanings remain viable as long as they continue to enable an individual to organize her experiences in a way that is internally consistent. In the case that a particular meaning and experience leads to a perturbation, an accommodation or reorganization is then necessary to reconcile this perturbation.

### Graphing, Shape Thinking, and Quantitative Reasoning

Stemming from a study exploring calculus students' meanings for two-variable functions, Weber (2012) characterized shape thinking as, "an association the student makes with a function's graph. For example, a student might associate a function's graph with a particular shape with physical properties while another student might associate a function's graph with a representation of quantities' values" (p. 17). In the present work, we use the phrase *shape thinking* to refer to this first aspect in which an individual's meaning for a graph is inferred directly from the pictorial image and perceptual properties of the physical shape. In contrast to this form of reasoning, Weber deemed activity rooted in reasoning about covarying quantities as *expert shape thinking*. For clarity reasons and to avoid implying that shape thinking is necessarily developmental, we do not use the phrase shape thinking to refer to the act of conceiving a graph as an emergent representation of how two quantities vary in tandem.

To provide contrasting examples of shape thinking and reasoning about covarying quantities, consider a student tasked with determining the formula for the graph in Figure 1.

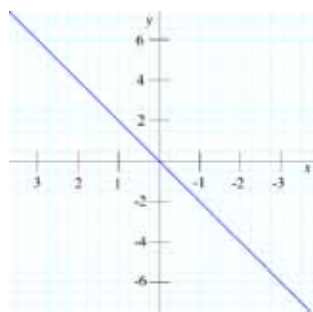


Figure 1: A Graph of  $y = 2x$

When providing a formula for the above graph, a shape thinker may first associate the graph with a formula of the form  $y = mx + b$  because she associates  $y = mx + b$  as defining a line. From there, she may conclude that  $m = -1$  by reasoning that  $m$  represents the tilt of the line and the line is downward sloping at a 45 degree angle with the horizontal axis. Lastly, she concludes that  $b = 0$  because the graph passes through the origin. In such a solution (e.g.,  $y = -x$ ), the students' activity foregrounds previously defined properties that are tied to perceptual attributes of the shape (e.g., straightness and tilt).

To provide a contrast to the above solution, another student may approach the problem by identifying that for each point captured by the curve, the  $y$ -value is two times as large as the  $x$ -value, leading the student to conclude that  $y = 2x$ . As another example, the student may determine that the curve captures

paired values such that for any change in  $x$ , the change in  $y$  is two times as large, and since the initial value (e.g., when  $x = 0$ ) for  $y$  is 0, then  $y = 2x$ . Whereas a shape thinker focuses on more global and pre-defined properties of the graph, essentially conceiving the graph “all at once” to determine the associated formula, these latter two solutions are generative in that the graph and its associated formula are conceived in terms of a dynamic relationship between quantities’ values; the latter students’ activities foreground a dynamic projection of points onto two axes to represent the quantities’ values.

We agree with Weber’s (2012) stance that shape thinking is not entirely bad. Shape thinking can support an individual in quickly inferring different representations and properties of a relationship conveyed by a graph (e.g., a line in the Cartesian coordinate system conveys a relationship that is also represented by  $y = mx + b$ ). But, a problem arises when shape thinking is in the absence of underlying meanings tied to inferences about quantities that vary in tandem (Goldenberg et al., 1992; Oehrtman et al., 2008; Thompson, 1994; Weber, 2012).

### Methodology

In order to better understand the nature and implications of shape thinking, we conducted semi-structured clinical interviews (Goldin, 2000) with ten undergraduate PSTs at a large university in the southeastern United States. We chose the PSTs on a voluntary basis. Each interview lasted approximately 90 minutes and all participants were given the same set of interview tasks. The PSTs were in their third undergraduate year, had already completed a minimum of two semesters of calculus plus two courses past the calculus sequence, and were currently enrolled in a functions and modeling course for PSTs. The course formed their first content course in the pre-service secondary mathematics teacher education program. We videotaped, transcribed, and then analyzed the interviews using conceptual analysis techniques (Steffe & Thompson, 2000) with a goal of characterizing the ways of thinking and meanings that supported their activity on the tasks.

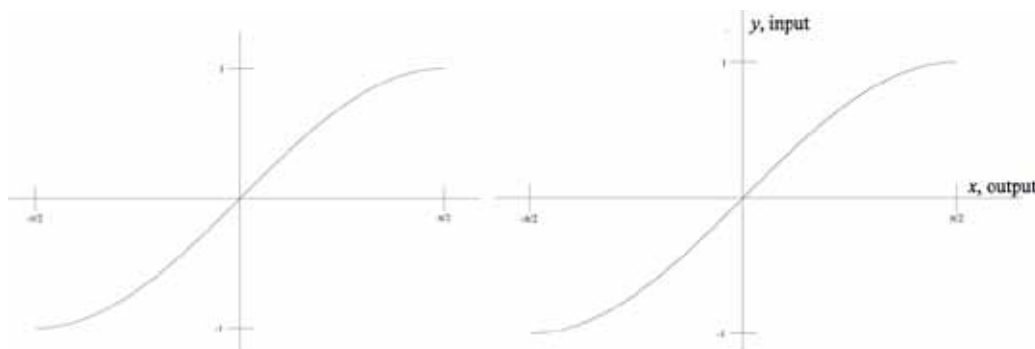


**Figure 2: Debating Graphs**

The choice of PSTs for this investigation was influenced by a variety of factors. First, our focus on shape thinking arose out of a classroom event during a previous year in the functions and modeling course. This event consisted of a debate that ensued as a group of PSTs argued whether two graphs with different orientations represented the same relationship (Figure 2). For some, the orientation of the axes and visual features of the graphs were significant components of their thinking leading them to maintain that the two graphs did not represent the same relationship; to these PSTs, the two graphs were visually different and thus had to represent different relationships. Based on this event, we anticipated that a subset of the following semester’s PST cohort would rely upon shape thinking. Further, we chose to work with PSTs because their meanings matter in that they influence their teaching practices and their future students’ learning (Simon, 2006; Thompson, in press). We hoped to better understand their meanings and subsequently improve teacher preparation based upon our understandings of their meanings. For instance, better understanding shape thinkers’ meanings may provide insights into how to perturb their meanings and generate learning opportunities.

### Task Design Example

When characterizing meanings, it's important that the researcher identify constraints in an individual's ways of operating, as these constraints aid the researcher in building more viable models of students' mathematics (Steffe & Thompson, 2000). In order to gain deeper insights into PSTs' meanings, we designed numerous tasks such that a shape thinker might encounter such constraints. As shape thinking is rooted in reasoning about pictorial objects in and of themselves, meanings rooted in shape thinking require that the situation conform to conventions upon which such meanings were abstracted. Thus, in our design of the tasks, we designed several non-canonical situations in order to better understand shape thinking by identifying how deep-rooted shape thinking may be, characterizing constraints engendered by shape thinking, and uncovering other meanings that that the PSTs used to reconcile problematic situations.



**Figure 3: A Task to Investigate Shape Thinking**

To illustrate one such task, consider the two graphs in Figure 3. The task proceeds: (i) first a PST determines a graph of the inverse sine function when given a graph of the sine function; (ii) the PST is then asked to comment on a graph (Figure 3, left) and prompt claiming that a student produced such a graph as an answer to part (i); (iii) lastly, the PST is presented with and asked to comment on a modified version (Figure 3, right) of the previous graph and a student explanation. The narrative provided with (iii) is, “*Well, because we are graphing the inverse of the sine function, we just think about  $x$  as the output and  $y$  as the input.*” When giving this explanation, the student added the following labels to their graph. (*Student quote in italics.*)

We designed parts (ii)-(iii) to provide information about the PSTs' meanings in the context of a situation that was non-canonical (e.g., considering axes as simultaneously representing input and output quantities of two functions). To a shape thinker, a function's name and a particular pictorial object are inherently connected. For this reason, we conjectured that the solutions in parts (ii)-(iii) might be problematic to a shape thinker because each presents one curve as conveying *two* functions,  $y = \sin(x)$  and  $x = \arcsin(y)$  for  $-\pi/2 \leq x \leq \pi/2$ .

### Results

As conjectured, the task described in the prior section revealed a number of instances that were suggestive of shape thinking. As a first example, consider Beth's (names are pseudonyms) response when determining a graph of the inverse sine function for part (i) of the task.

*Beth:* I should know this. Umm...I mean...*(pauses for 30 seconds and writes  $y = \sin(x)$ ).*

*Int:* So how are you thinking about finding the inverse of that? Or the graph of the inverse?

*Beth:* *(Writes  $\sin^{-1} y = x$ ).* So I mean, obviously it's just *(writes  $\sin^{-1} x = y$ )* which doesn't help at all. *(Laughs).* Umm, but, I'm thinking that it's either going to be something like...actually I'm kind of embarrassed because I really should know this.

In this exchange, Beth first attempted to recall a shape from memory (e.g., “I should know this”), an approach that was not successful. Beth then symbolically manipulated  $y = \sin(x)$  into  $\sin^{-1} x = y$ , but obtaining this formula was not satisfying to Beth because her difficulty still lied within attempting to remember the shape of the graph (e.g., “I really should know this”). Following the above exchange, Beth made a few conjectures regarding the shape of the inverse sine graph, but as the discussion continued it became apparent that these conjectures were attempts to recall the graph from memory. Beth explained that she could not confidently determine an answer. We then progressed to parts (ii) and (iii) of the task to see how Beth would make sense of a posed solution to the problem.

*Int:* So a student came to you with that graph (*Figure 3, left*) claiming that it was a graph of the inverse sine function. What would you say to that? Could that be true?

*Beth:* Could that be true? Umm...(*thinks silently for 15 seconds*)...mmm, no. No.

*Int:* Why are you thinking no?

*Beth:* ...I'm thinking this just kind of looks like...sine graph – like the plain sine graph. (*Laughs*). Which is going to be different. So, no...

(*Interviewer later gives Beth part (iii), Figure 3, right*).

*Beth:* I mean I guess what I'm like thinking about, like struggling with thinking is that like, like I don't know if, or if...an inverse function...like the graph of an inverse function, like, can't be the same as the original graph. Or can it? Like, I wouldn't think that it could. But maybe there's something I don't know. (*Sighs*).

Beth's responses and uncertainty in parts (ii) and (iii) are viable when viewed through the lens of shape thinking. Given this perspective, two different functions cannot be represented by the same graph because otherwise they would be the same function; a shape or curve has only one associated function name as a label. Thus, even though Beth was willing to admit there may have been something she did not know, she was unable to determine a way in which a student might conceive the graph as both the sine function and sine inverse function.

While Beth's perturbations appear to stem from her belief that different functions must have different shapes, another PST experienced conflict for apparently different reasons when responding to part (iii).

*Megan:* Oh, Um, Um. We just can't do that (*in response to Figure 3, right*).

*Int:* OK, say a little bit more.

*Megan:* (*Laughs*). It's not, that's not, his definition is not necessarily wrong. But you can't just label it like that. Um, why? Why can't you do that? I don't know, I feel like he's missing the whole concept of a graph...Like, I know you can call whatever axis you know if you are doing time and weight or volume whatever...But not necessarily with the sine graph. Like the sine graph's like a...it's a graph that everyone knows about...They are just missing the concept of graphing.

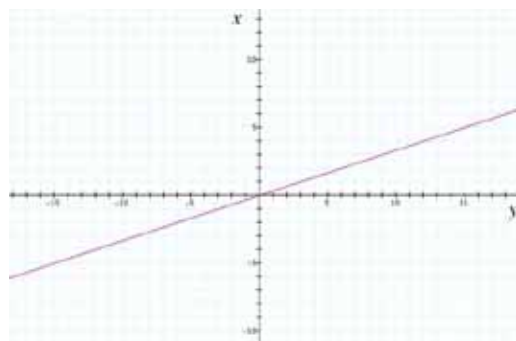
Whereas we did not identify an instance in which Beth focused on input and output quantities, Megan identified that the posed graph involved denoting the axes in various ways relative to input and output quantities. Megan seemed willing to accept reversing the roles of the input and output for some quantities such as time and weight, but not in the case of the given graph. We believe that her comment, “it's a graph everyone knows about” indicates that for the given graph, the input and output axes are fixed because that particular curve is designated as the sine function. Hence, changing the roles of the axes in the provided graph produces a conflict.

We see that both Megan and Beth experienced unique perturbations when faced with a non-standard approach to graphing the inverse sine function. Yet, underlying both perturbations is a similar phenomenon that rests in their uniquely associating the given curve with the sine function. Due to the nature of this association, the given curve could not also be considered as the inverse sine function, even in the case that a PST (e.g., Megan) acknowledged the possibility of considering different axes for the input and output quantities. In each case, because the curve was “known” by a particular label, it was not possible to be given another label.

Another form of shape thinking we observed was instances in which the PSTs' reasoning was focused on the global properties of a curve in ways that were not attentive to emergent quantitative



relationships. For example, Beth was presented with the following hypothetical student's graph of  $y = 3x$  (note the non-standard orientation of the  $x$  and  $y$  axes in Figure 4).



**Figure 4: A Hypothetical Student's Non-standard Graph of  $y = 3x$**

*Int:* How would you respond to the student or how might the student be thinking about this?

*Beth:* Umm...like this (*spinning paper 90 degrees counterclockwise*). (*Laughs*). Like, because if you turn it this way then this (*traces left to right along the x-axis which is now in the horizontal position*) and this (*traces top to bottom along the y-axis*) and it would be still not right though (*spinning paper back to original orientation*).

*Int:* And how would you respond to this student if they said, "Well here's how I'm thinking?"

*Beth:* I guess...I mean the only way I can think of it is like this (*spinning paper 90 degrees counterclockwise*) and it's still wrong because this (*a line which is now sloping downward left to right*) is negative slope. So I would just, I would just explain to them, like the difference between the  $x$ - and  $y$ -axes and umm ...show them like the difference between positive and negative slopes also. Because that's something that, like, when I was in middle school we, like, learned kind of like a trick to remember positive, negative, no slope, and zero (*making hand motions to indicate each*). Like where the slopes were. And it's stuck with me until now so it's important to know which direction they're going...

Beth's actions convey several instances of shape thinking. First, Beth's insistence that the line has a negative slope after rotating the paper 90 degrees counterclockwise to orient the  $x$ -axis horizontally and her reference to remembering slope-curve orientation pairs provides evidence that her meanings for slope are connected to a particular shape (e.g., a line sloping downward left to right implies negative slope). Beth's repeated moves to reorient the graph and her later rejection of the hypothetical student's solution is also evidence of her reliance on shape thinking. Beth's rotating of the graph was an attempt to achieve a conventional axes orientation, which would enable judging the solution against her image of the proper graph. Because Beth was unable to achieve an orientation compatible with her image of the appropriate graph, she maintained that the graph was "wrong" and described that she was not sure what the student did.

Although shape thinking was a common way of reasoning for many of the PSTs throughout the interviews, a few of the participants utilized other ways of reasoning to make sense of the given situations. For instance, in contrast to Beth's interpretations of the non-standard graph of  $y = 3x$ , another PST, Jacob, did not rely upon shape thinking. He instead made sense of the hypothetical student's work by conceiving the graph in terms of a quantitative relationship. After taking a few moments to inspect the graph (Figure 4, axes unlabeled), Jacob decided to consider a few of the paired values. This led him to conclude that the student was "plugging in" values on the  $y$ -axis (e.g., consider  $x$  from the formula as along the typical  $y$  axis). After seeing the student's work with the vertical axis being labeled as  $x$ , Jacob accepted the student's work as correctly conveying the relationship  $y = 3x$ . Although this type of

quantitative reasoning was rarely observed during the interviews, we include it as a contrast to the other results in order to highlight the implications of relying upon shape thinking as one makes sense of graphs.

### Discussion and Concluding Remarks

Over the course of the interviews, the PSTs predominantly engaged in shape thinking, which is unsurprising given the body of literature documenting meanings of this nature (e.g., Thompson, 1994, in press; Weber, 2012). What was unforeseen was how the PSTs' meanings influenced their activity on problems that problematized shape thinking. The confusions the PSTs faced during the interview tasks provide insights into the possible genesis of shape thinking. Instances of shape thinking became apparent when the given interview tasks broke common practices or conventions in school mathematics (e.g., axes orientation). As such, the PSTs' struggles can be partly framed as a consequence of constructing meanings inherently tied to these conventions and encountering interview tasks that did not conform to the conventions from which shape thinking was abstracted. The PSTs' propensity to engage in shape thinking highlights that such thinking had become an integral part of these PSTs' mathematical meanings (e.g., meanings for the sine function, for inverse, and for slope). It follows that they likely had repeated experiences in which to (re)construct these meanings as viable. In other words, by repeatedly encountering situations that conformed to particular conventions, they abstracted meanings that were inherently tied to these conventions. For example, Beth's slope meanings were tied to shapes reliant on a particular axes orientation that is pervasive in mathematics.

Obviously mathematical conventions are critical supports to mathematical activity. But, a problem arises when meanings are tied to these conventions in ways that restrict one's ability to reconcile situations that are internally consistent but do not follow such conventions. For instance, the graph in Figure 4 is a quantitatively correct representation. Yet, several PSTs were not able to provide a viable explanation for the student's proposed graph because their meanings required that the graph be in a more conventional orientation. Such an outcome speaks to a possible drawback of mathematical conventions in the context of student learning; by repeatedly experiencing situations that conform to particular conventions, these conventions can become constraining aspects of student meaning. Then, as these students later become teachers, the conventions constrain how they may operate with their students and interpret student work.

Glaserfeld (1995) stated, "Actions, concepts, and conceptual operations are viable if they fit the purposive or descriptive contexts in which we use them" (pg. 14). The prevalence and deep-rooted nature of shape thinking among the PSTs in the study implies that their meanings had consistently been "purposive and descriptive" during their previous mathematical experiences. By breaking from common graphing conventions, the tasks in this study created situations in which shape thinking did not provide a viable solution. This problematized the PSTs' meanings, which would have generated teaching and learning moments in the classroom or another setting (e.g., a teaching experiment). Further research should explore how such tasks could be used to support students' development of meanings that entail reasoning with quantitative relationships. Additionally, further research is necessary to characterize possible implications of particular conventions in K-12 mathematics in the context of student learning.

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